Initial Motions of a Jacobi ellipsoid at the point of bifurcation

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1. Introduction

In discussing the figure of the Earth, I. Newton assumed that it is a spheroid with the axis of symmetry coinciding with the axis of rotation. Since then. it was shown rigorously by Maclaurin that a spheroid is a possible form of equilibrium of incompressible rotation fluid for a give value of angular velocity. Later, Jacobi showed that an ellipsoid of three different axes is also a possible form of equilibrium, provided that the angular velocity exceeds a certain value. In all of these investigations, the fluid mass is assumed having uniform rotation. In all of these studies, the angular velocity enters into the equations governing the forms through the factor $\omega^2/2\pi G\rho$, where ω, ρ and G denote the angular velocity, density (assumed constant) and the constant of gravity, respectively. It follows that when one follows the evolution of a fluid mass with increasing density, it would be equivalent to assuming increasing angular momentum, and we will assume so throughout the present investigation, as in previous work.

As mentioned, Jacobi ellipsoids are possible forms of equilibrium provided that the angular momentum exceeds a certain value. Later, H. Poincaré (1885) and G. Darwin (1886) showed that the so called pear shaped figure bifurcates when the angular velocity of the Jacobi ellipsoid exceeds a certain value. By an extensive investigation, it was shown by Jeans (1929) that the pear-shaped figure are secularly unstable. Since then, Cartan (1926) showed that the Jacobi ellipsoid becomes ordinarily unstable when the pear-shaped figure bifurcates. Chandrasekhar & Lebovitz (1963) used the tensor virial method and Yabushita (1965) used the method suggested by Cartan (1926) to calculate the time constants of the unstable motion away from the ellipsoids.

The object of the present paper is to investigate the initial value problem associated with the bifurcating mode, in general, and to show that there is no ambiguity in specifying the initial motions of the fluid mass. It will further be shown that the departure of the pear shaped figure from the critical Jacobi ellipsoid proceeds as At + B, where t is the time and A and B are constants specified by the initial conditions.

2. Equations of motion and the surface condition

In the Lagrangian method of fluid dynamics we are interested in the motion of a fluid element $\delta x \ \delta y \ \delta z$, which at time $t = 0$ occupies a point (x, y, z). Let the coordinates of the element $\delta x \ \delta y$

δz at time, t be X, Y, Z. Apparently the position of a fluid element is a function of x, y, z and t:

$$X = X\ (x, y, z, t),$$
$$Y = Y\ (x, y, z, t),$$
$$Z = Z\ (x, y, z, t).$$

The equations of motion for the fluid referred to the coordinate system which is rotating around the axis with angular velocity ω are:

$$\frac{\partial^2 X}{\partial t^2} - 2\omega \frac{\partial Y}{\partial t} - \omega^2 X = \frac{\partial \phi}{\partial X} - \frac{1}{\rho}\frac{\partial p}{\partial X},$$

$$\frac{\partial^2 Y}{\partial t^2} + 2\omega \frac{\partial Y}{\partial t} - \omega^2 Y = \frac{\partial \phi}{\partial Y} - \frac{1}{\rho}\frac{\partial p}{\partial Y},$$

$$\frac{\partial^2 Z}{\partial t^2} = \frac{\partial \phi}{\partial Z} - \frac{1}{\rho}\frac{\partial p}{\partial Z},$$

where ϕ is the gravitational potential, ρ is the density of the fluid, and p is the pressure.

To deduce the equations containing only derivatives with respect to the independent variables x, y, z, t, we multiply the above equations by

$$\frac{\partial X}{\partial x},\ \frac{\partial Y}{\partial y},\ \frac{\partial Z}{\partial z}$$

and add, obtaining

$$\frac{\partial^2 X}{\partial t^2}\frac{\partial X}{\partial x} + \frac{\partial^2 Y}{\partial t^2}\frac{\partial Y}{\partial x} + \frac{\partial^2 Z}{\partial t^2}\frac{\partial Z}{\partial x} + 2\omega\left(\frac{\partial X}{\partial t}\frac{\partial Y}{\partial x} - \frac{\partial Y}{\partial t}\frac{\partial X}{\partial x}\right)$$
$$- \omega^2\left(X\frac{\partial X}{\partial x} + Y\frac{\partial Y}{\partial x}\right) = \frac{\partial \phi}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

and similar equations containing derivatives with respect to y and z. The equation of continuity for uniform density is

$$\frac{\partial (X, Y, Z)}{\partial (x, y, z)} = 0$$

The displacement of a fluid element from its original position is given by:

$$\xi\ (x, y, z, t) = X(x, y, z, t) - x,$$
$$\eta\ (x, y, z, t) = Y(x, y, z, t) - y,$$
$$\zeta\ (x, y, z, t) = Z(x, y, z, t) - z.$$

In the following we shall be concerned with small motions of a fluid around an equilibrium form and therefore ξ, η, ζ and their time derivatives will be regarded as small quantities of the first order. The equations of motion then take the following form;

$$\left.\begin{aligned} \frac{\partial^2 \xi}{\partial t^2} - 2\omega \frac{\partial \eta}{\partial t} &= \frac{\partial \chi}{\partial x} \\ \frac{\partial^2 \eta}{\partial t^2} + 2\omega \frac{\partial \xi}{\partial t} &= \frac{\partial \chi}{\partial y} \\ \frac{\partial^2 \xi}{\partial t^2} \qquad\qquad &= \frac{\partial \chi}{\partial y} \end{aligned}\right\} \tag{2.1}$$

where

$$\chi = \frac{p}{\rho} - \phi - \frac{\omega^2}{2}(x^2 + y^2). \tag{2.2}$$

The equation of continuity takes the form

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0 . \tag{2.3}$$

In any configuration of relative equilibrium, relative to the rotating frame, $\ddot{\xi}$, $\ddot{\eta}$, $\ddot{\zeta}$, $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ must vanish and so we get

$$\frac{\partial \chi}{\partial x} = \frac{\partial \chi}{\partial y} = \frac{\partial \chi}{\partial z} = 0 ,$$

or

$$\chi = \frac{p}{\rho} - \phi - \frac{\omega^2}{2}(x^2 + y^2) = \chi_0 , \tag{2.4}$$

where χ_0 is not only independent of x, y, z, but also independent of t. Now we investigate the properties of the solution of the equations of motion (2.1) supplemented by the equation of continuity (2.3). Eqns. (2.1) may be regarded as a set of partial differential equations for χ, and in order to solve them we have to specify boundary conditions to be satisfied by χ. We shall not derive the well-established results for the surface condition for χ, but will present them without detailed proof. The readers interested in this are referred to Lyttleton (1953).

It is convenient to express χ as

$$\chi(x, y, z, t) = \psi(x, y, z, t) + \chi_0 , \tag{2.5}$$

where χ_0 is the value of χ for the equilibrium configuration, and is, by (2.4), constant. It can be shown that if σ, defined by

$$\sigma \equiv \frac{x}{a^2}\xi + \frac{y}{b^2}\eta + \frac{z}{c^2}\zeta = 1 \tag{2.6}$$

is expanded on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

in terms of Lamé surface harmonic functions

$$\sigma = \sum_k A_k \; M_k \; N_k, \tag{2.7}$$

then, on the ellipsoid, ψ must take the form

$$\psi = 2\pi \; G\rho \; \sum_k (H_o - H_k) \; A_k \; M_k \cdot N_k \, , \tag{2.8}$$

where $M_k \cdot N_k$ are Lamé functions evaluated at the surface of the ellipsoid, and $H_o - H_k$ are the so-called coefficients of stability defined by

$$H_o = \frac{2}{3} abc \left\{ L_1(\lambda) \; S_1(\lambda) \right\}_{\lambda=0} ,$$

$$H_k = \frac{2abc}{2n+1} \left\{ L_k(\lambda) \; S_k(\lambda) \right\}_{\lambda=0} ,$$

$L_k(\lambda)$, $S_k(\lambda)$ being Lamé function of the first and second kind, respectively. $L_1(\lambda)$ and $S_1(\lambda)$ are particularly defined by

$$L_1(\lambda) = (\lambda + c^2)^{1/2},$$

$$S_1(\lambda) = (c^2 + \lambda)^{1/2} \int_\lambda^\infty \frac{3\, d\lambda}{2\left[(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)^3\right]^{1/2}} \, ,$$

and $S_k(\lambda)$ is derived from $L_k(\lambda)$ by

$$S_k(\lambda) = L_k(\lambda) \int_\lambda^\infty \frac{(2n+1)\, d\lambda}{2 L_k^2(\lambda)\left[(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\right]^{1/2}} \, ,$$

where n is the order of $L_k(\lambda)$. It should be noted that a constant multiple of a Lamé function is also a Lamé function, and a multiplicative factor can always be chosen arbitrarily. H_o and H_k are, however, unaltered by these multiplicative factors, as can be see from the above definitions.

The $(H_o - H_k)$ are usually called the 'coefficients of stability,' for when the difference of the total energy (gravitational and centrifugal) of a distorted ellipsoid and that of the original ellipsoid is expressed as a homogeneous quadratic form in A_k's, the coefficients of A_k^2 are constant multiples of $H_o - H_k$. Therefore if all the coefficients $H_o - H_k$ are positive the total potential energy is minimum when the ellipsoid is undistorted, and therefore the ellipsoid is a secularly stable configuration. On the other hand, if some of $H_o - H_k$ were negative the ellipsoid would no longer be a configuration of minimum potential energy and would no longer be a secularly stable configuration. If a liquid mass is regarded as evolving along the series of the Jacobi ellipsoids with gradually increasing angular momentum the ratio of the axes a : b : c slowly changes and so do $H_o - H_k$. Since there are $2n+1$ Lamé functions of order n, there are $2n+1$ coefficients of stability. It has been proved by Poincaré that as the ratio a:b:c gradually changes in the direction of increasing angular momentum, one of the coefficients of stability of order 3 first changes sign, and then one of the coefficients of stability of order 4 changes sign and so on. For any order n, only one of $2n+1$ coefficients of stability changes sign and, in particular, the Lamé functions of order 3 through which the secular stability sets in is

$$L\ (\lambda) = (\lambda + a^2)^{1/2}\ (\lambda + h)\ ,$$

which corresponds to

$$LMN \propto \quad x \left[\frac{x^2}{a^2-h} + \frac{y^2}{b^2-h} + \frac{z^2}{c^2-h} - 1 \right]$$

where

$$h = \frac{1}{5}\ (a^2 + 2b^2 + 2c^2) + \frac{1}{5}\ (a^4 + 4b^4 + 4c^4 - 7b^2c^2 - c^2a^2 - a^2b^2)^{1/2}.$$

The ratio of the axes of the critical Jacobi ellipsoid has been calculated by Darwin (1886) who gives $a : b : c : (abc)^{1/3} = 1.88583 : 0.81498 : 0.65066 : 1$,

$$\frac{\omega^2}{2\pi G \rho} = 0.14200\ .$$

All the ellipsoids beyond this point are secularly unstable.

3. The free oscillations of order n.

In the present section we investigate how the fluid motion is determined in terms of initial conditions. We have first the equations of motion

$$\left.\begin{aligned} \frac{\partial^2 \xi}{\partial t^2} - 2\omega \frac{\partial \eta}{\partial t} &= \frac{\partial \psi}{\partial x}, \\ \frac{\partial^2 \eta}{\partial t^2} + 2\omega \frac{\partial \xi}{\partial t} &= \frac{\partial \psi}{\partial y}, \\ \frac{\partial^2 \xi}{\partial t^2} \qquad\qquad &= \frac{\partial \psi}{\partial z}, \end{aligned}\right\} \tag{3.1}$$

and the equation of continuity

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0. \tag{3.2}$$

In order to express the surface condition for ψ in aconvenient form we first notice that Lamé functions are, on the ellipsoid, orthogonal to each other;

$$\iint M_k N_k M_\ell N_\ell\, p\, dS = 0\ , \qquad \text{if } k \neq \ell$$

where

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$$

and dS is a surface element of the ellipsoid. Eliminating A_k from (2.7) and (2.8) using the above orthogonality relation we get;

$$\iint M_k N_k \, \psi \, p \, dS = 2\pi G \rho (H_o - H_k) \iint M_k N_k \left(\frac{x}{a^2}\xi + \frac{y}{b^2}\eta + \frac{z}{c^2}\zeta \right) p dS . \qquad (3.3)$$

It should now be noticed that if we put

$$\psi(x, y, z, t) = \psi_n(x, y, z, t) + \psi_{n-1}(x, y, z, t) + \ldots$$
$$\xi(x, y, z, t) = \xi_{n-1}(x, y, z, t) + \xi_{n-2}(x, y, z, t) + \ldots$$
$$\eta(x, y, z, t) = \eta_{n-1}(x, y, z, t) + \ldots$$
$$\zeta(x, y, z, t) = \zeta_{n-1}(x, y, z, t) + \zeta_{n-2}(x, y, z, t) + \ldots$$

where ψ_n is a homogeneous polynomial of degree n in x, y, z, and so on, the equations for the homogeneous parts $\psi_n(x, y, z, t)$ and $\xi_{n-1}(x, y, z, t), \eta_{n-1}(x, y, z, t), \zeta_{n-1}(x, y, z, t)$ do not contain any of the lower polynomials. Therefore as long as the motions of order n are concerned, $\psi(x, y, z, t)$ and $\xi(x, y, z, t)$, etc. may be regarded as consisting of homogeneous polynomials of order n and n − 1 respectively. We put:

$$\xi = \Sigma\, \alpha_{pqr}(t) x^p y^q z^r , \quad p+q+r = n-1 ,$$
$$\eta = \Sigma\, \beta_{pqr}(t) x^p y^q z^r , \quad p+q+r = n-1 ,$$
$$\zeta = \Sigma\, \gamma_{pqr}(t) x^p y^q z^r , \quad p+q+r = n-1 ,$$
$$\psi = \Sigma\, \Phi_{pqr}(t) x^p y^q z^r , \quad p+q+r = n .$$

ξ, η, ζ and ψ will uniquely be determined if α, β, γ and Φ are determined. The number of the Φ's in ψ is $\frac{1}{2}(n+1)(n+2)$, the number of α's being $\frac{1}{2}n(n+1)$, etc., in total we have $2n^2 + 3n + 1$ $(= \frac{3}{2}n(n+1) + \frac{1}{2}(n+1))$ coefficients to be determined. On the other hand each equation of motion gives $\frac{1}{2}n(n+1)$ relations, the equation of continuity provides $\frac{1}{2}n(n+1)$ relations, and the surface condition (3.3) provides $2n+1$ relations available to determine α, β, γ and Φ. Therefore the number of the unknowns is equal to the number of equations, there being no ambiguity at all.

In order to obtain the frequencies of the free oscillations of the liquid mass, we further put

$$\alpha_{pqr}(t) = \alpha_{pqr} e^{i\lambda t}, \quad \beta_{pqr}(t) = \beta_{pqr} e^{i\lambda t}, \quad \gamma_{pqr}(t) = \gamma_{pqr} e^{i\lambda t}$$
$$\Phi_{pqr}(t) = \Phi_{pqr} e^{i\lambda t}$$

where α, β, γ and Φ are constants. By comparing the coefficients of $e^{i\lambda t}$ we get first from the equations of motion

$$\left.\begin{aligned} -\lambda^2 \alpha_{pqr} - 2i\omega\lambda\beta_{pqr} &= (p+1)\, \Phi_{p+1,q,r} \\ -\lambda^2 \beta_{pqr} + 2i\omega\lambda\alpha_{pqr} &= (q+1)\, \Phi_{p,q+1,r} \\ -\lambda^2 \gamma_{pqr} \qquad\qquad &= (r+1)\, \Phi_{p,q,r+1} \end{aligned}\right\} \qquad (3.4)$$

and from the equation of continuity,

$$p\,\alpha_{pqr} + (q+1)\,\beta_{p-1,q+1,r} + (r+1)\,\gamma_{p-1,q,r+1} = 0 , \qquad (3.5)$$

and from the surface condition (3.3),

$$2\pi G\rho\,(H_o - H_k) \times (\text{linear combination of } \alpha\text{'s},\ \beta\text{'s},\ \gamma\text{'s}) = \text{linear combination of } \Phi\text{'s} \tag{3.6}$$

$k = 1, 2, \ldots\ldots 2n+1$.

The above is a set of linear homogeneous equations. In order to put these into a matrix form it is convenient to arrange α , β , γ and Φ in the following way:

$$\begin{array}{ll}
\alpha_{pqr} = \omega_m & 1 \leqq m \leqq \frac{1}{2} n\,(n+1) \\
\beta_{pqr} = \omega_m & \frac{1}{2} n\,(n+1) + 1 \leqq m \leqq n\,(n+1) \\
\gamma_{pqr} = \omega_m & n\,(n+1) + 1 \leqq m \leqq \frac{3}{2} n\,(n+1) \\
\Phi_{pqr} = \omega_m & \frac{3}{2} n\,(n+1) + 1 \leqq m \leqq 2n^2 + 3n + 1\ .
\end{array}$$

Then (3.4), (3.5) and (3.6) can be put into the form

$$\mathrm{M} \cdot \underset{\sim}{\omega} = 0 \tag{3.7}$$

where $\underset{\sim}{\omega}$ is the vector $(\omega_1, \omega_2, \ldots\ldots\ldots, \omega_{2n^2+3n+1})$ and M is a matrix;

$$M \equiv \left(\begin{array}{c:c:c:c}
\begin{matrix}\lambda^2 & & \\ & \ddots & \\ & & \lambda^2\end{matrix} & \begin{matrix}\lambda & & \\ & \ddots & \\ & & \lambda\end{matrix} & \begin{matrix}0\ 0 & \\ & \\ 0\ 0\ \ldots & 0\end{matrix} & \begin{matrix}0,\ 0, & \\ & \\ 0,\ 0,\ \ldots & 0\end{matrix} \\ \hdashline
\begin{matrix}\lambda & & \\ & \ddots & \\ 0,\ldots & & \lambda\end{matrix} & \begin{matrix}\lambda^2 & & \\ & \ddots & \\ 0,\ldots & & \lambda^2\end{matrix} & \begin{matrix}0,\ldots & 0 \\ & \\ 0 & \end{matrix} & \begin{matrix}0, & 0 \\ & \\ 0,\ldots & 0\end{matrix} \\ \hdashline
\begin{matrix} & \\ & \\ 0,\ldots & 0\end{matrix} & \begin{matrix} & \\ & \\ 0,\ldots & \end{matrix} & \begin{matrix}\lambda^2 & & \\ & \ddots & \\ 0 & & \lambda^2\end{matrix} & \begin{matrix} & \\ & \\ 0,\ldots & 0\end{matrix} \\ \hdashline
NON & ZERO & NON\ ZERO & \begin{matrix}0,\ldots\ldots\ldots\ldots\ldots & \\ \ldots & 0 \\ 0,\ldots\ldots\ldots\ldots\ldots & \\ \ldots & 0\end{matrix} \\ \hdashline
\multicolumn{3}{l:}{\begin{matrix}(H_o - H_1) \\ (H_o - H_2) \\ \\ (H_o - H_{2n+1})\end{matrix}} & NON\text{-}ZERO
\end{array}\right)
\begin{array}{l}
\frac{n}{2}(n+1) \\ \\ \frac{n}{2}(n+1) \\ \\ \frac{n}{2}(n+1) \\ \\ \frac{n}{2}(n-1) \\ \\ 2n+1
\end{array} \tag{3.8}$$

In the above expression for the matrix M, factors independent of λ are not explicitly written down. The first $\frac{3}{2}n(n+1)$ rows come from eq. (3.4) and the next $\frac{1}{2}n(n-1)$ rows from eq. (3.5). The last $2n+1$ rows come from eq. (3.6) and, each of the first $\frac{3}{2}n(n+1)$ elements of the first row in this group contains $(H_o - H_1)$ as a multiplicative factor, each of the second $\frac{3}{2}n(n+1)$ elements contains $(H_o - H_1)$, and so on. In order that non-zero solutions of (3.7) may exist, the determinant of the matrix M must vanish;

$$D(\lambda) \equiv \text{determinant of } M = 0. \tag{3.9}$$

This is the equation which determines permissible values of λ.

4. The number of free oscillations of order n.

We start this section by discussing some properties of determinants which are relevant when we consider the number of free oscillations of order n. First we consider an N x N determinant of the form

$$d(\lambda) \equiv \begin{array}{r|l|l} & \underbrace{\boxed{\begin{array}{l}\text{Each element in this block} \\ \text{contains at least one } \lambda \text{ as a} \\ \text{multiplicative factor,}\end{array}}}_{L} & N \\ M\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & & \end{array}$$

where (i) $L > M$, (ii) each element in the first Lx $(N-M)$ block contains at least one λ as a multiplicative factor, and (iii) none of the elements in the other block contains λ. First of all we notice that when $d(\lambda)$ is regarded as a polynomial in λ, the lowest power of $d(\lambda)$ is at least $L-M$. Again, if λ is put equal to zero in $d(\lambda)$ we can easily see that all the first, second minor determinants up tp $(L-M-1)$ th minor determinants of d (O) are zero and there is only one $(L-M)$ th minor determinant of d (O) which is different from zero. This non zero $(L-M)$ th minor determinant is the coefficient of $\lambda^{L\text{-}M}$ when d (λ) is expressed as a polynomial in λ.

We now apply the results obtained above to the M defined by (3.8).

We see that the determinant $D(\lambda)$ of M is a polynomial in λ of degree

$$2\left[2\frac{n}{2}(n+1) + n\right] = 2n^2 + 4n.$$

The lowest power of $D(\lambda)$ is at least λ^{n^2-1} and $D(\lambda)$ must have the form

$$D(\lambda) = \text{constant (which contains none of } (H_o - H_k)) \ \lambda^{2n^2+4n} +$$

$$+ \ldots\ldots \text{constant} \times \prod_{k=1}^{2n+1} (H_o - H_k)\lambda^{n^2-1} = 0. \tag{4.1}$$

Therefore $\lambda = 0$ is an $(n^2 - 1)$ ple root of $D(\lambda) = 0$, and the number of other roots

$$\text{const.} \times \lambda^{n^2+4n+1} + \ldots\ldots \text{const.} \times \prod_{k=1}^{2n+1} (H_o - H_k) = 0 \ , \tag{4.2}$$

is $n^2 + 4n + 1$. Since each root λ is paired with $-\lambda$, $\lambda = 0$ must be another root of (3.9) when n is even. In order that $\lambda = 0$ may be a root of (3.9) the constant term of (3.9) must be zero. Thus we come to the conclusion that

(a) When n is odd $D(\lambda) = 0$ has an $(n^2 - 1)$ ple root $\lambda = 0$ and the other roots are given by (3.10). The constant term in (3.10) is the only non-zero n^2 th minor determinant of $D(0)$, when $D(0)$ is regarded as a determinant.

(b) When n is even $D(\lambda) = 0$ has an n^2 ple root $\lambda = 0$ and D (0) has the property that when it is regarded as a determinant, all of the first, second minor determinants up to $(n^2 - 1)$ th minors are all zero.

5. Ordinary stability.

It is well known in the theory of differential equations that the general solution of a set of differential equations

$$\sum_j \left[a_{ij} \frac{d^2 A_j}{dt^2} + b_{ij} \frac{dA_j}{dt} + c_{ij} A_j \right] = 0$$

where a_{ij}, b_{ij}, c_{ij} are constant, has the form

$$A_i(t) = e^{\lambda t}(g_1 t^{\alpha-1} + g_2 t^{\alpha+2} + \ldots\ldots)$$

where $g_1, g_2, \ldots\ldots$ are constant, if the determinantal equation

$$\text{Det.} \left| a_{ij}\ \lambda^2 + b_{ij}\ \lambda + c_{ij} \right| = 0 \ ,$$

has a multiple root λ with multiplicity α, and in general a multiple root λ indicates an unstable motion. With regard to this we have a theorem due to Routh (1884).

'When α equal roots occur in the determinant, and the terms in the solution with t as a factor are to be absent, it is necessary as well as sufficient that all the first, second minors up to the $(\alpha - 1)$ th should be zero.'

As we have seen in the previous section $\lambda = 0$ is an n^2 ple root when n is even and an $(n^2 - 1)$ ple root when n is odd. However, the D (0) regarded as a determinant has the property that all the minors of D (0) up to the $(n^2 - 2)$ th minors when n is odd, and up to the $(n^2 - 1)$ th minors when n is even, are zero. Thus by the Routh's theorem the multiple root does not cause any instability of motion.

As far as the ordinary instability is concerned, the case $n = 3$ is of great interest, for the secular instability of the Jacobi series first enters through a surface deformation of order 3. From (3.8) we have for $n = 3$

$$\lambda_1^2 . \lambda_2^2 . \ldots\ldots \lambda_{11}^2 = \text{const.} \times \prod_{k=1}^{7} (H_o - H_k)$$

where $\lambda_1 \ldots \lambda_{11}$ are arranged so that $\lambda_1 \leqq \lambda_2 \leqq \ldots \lambda_{11}$.

When a liquid mass is in a configuration of a secularly stable ellipsoid it must necessarily be ordinarily stable and the right hand side of the above equation must be positive. As the liquid mass evolves along the Jacobi series only one of $(H_0 - H_k)$ changes sign and consequently λ_1^2 or λ_1^4, when $\lambda_1 = \lambda_2$, must change sign at the point where the secular stability is lost. This means that λ_1 must be purely imaginary or complex where the secular instability ceases and thus the ordinary instability of the Jacobi series sets in at the point where the secular stability is lost, a result due to Cartan. Mathematical discussion does not, however, allow us to decide whether λ_1 is imaginary or complex and the decision has to wait until the actual numerical calculations are carried out. This has been done by Chandrasekhar & Lebovitz (1963) and by Yabushita (1965). The λ has been found to be purely imaginary.

6. Consideration concerning initial conditions.

From physical consideration, the initial condition of the motion of a liquid mass can be specified by the values of $\xi(x, y, z, t)$, $\eta(x, y, z, t)$, $\zeta(x, y, z, t)$ and their first time derivatives at $t = 0$. ξ, η, ζ at $t = 0$ are not entirely independent, for they must satisfy the equation of continuity

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0 , \qquad t = 0 . \tag{6.1}$$

A further restriction comes from the equation of continuity differentiated with respect to t.

$$\frac{d}{dt}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right) = 0 \quad \text{or} \quad \frac{\partial \dot{\xi}}{\partial x} + \frac{\partial \dot{\eta}}{\partial y} + \frac{\partial \dot{\zeta}}{\partial t} = 0 , \quad \text{at} \quad t = 0 . \tag{6.2}$$

The condition (6.1) will not need further explanation. To show that (6.2) does restrict the initial velocity $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, we consider the following example.

A particle is restricted to move along a plane curve,

$$f(x, y) = 0$$

in a given field of force. The equations of motion,

$$\ddot{x} = F(x, y)$$
$$\ddot{y} = G(x, y)$$

allow four arbitrary constants. Two of these, x_0 and y_0, the coordinates at $t = 0$ are not independent for they must satisfy the condition $f(x_0, y_0) = 0$. The initial velocities $\dot{x}_0$ $\dot{y}_0$ are again not independent for they must satisfy the condition

$$\frac{d}{dt} f(x, y) = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} = 0 , \qquad \text{at} \quad t = 0 .$$

Now, $\xi(x, y, z, t)$, $\eta(x, y, z, t)$, $\zeta(x, y, z, t)$ at $t = 0$ are uniquely specified by $\frac{3}{2} n(n+1)$ coefficients ($\frac{n}{2}(n+1)$ for a's in $\xi = \Sigma\, a_{pqr}\, x^p y^q z^r$) and the same is true for $\dot{\xi}(x, y, z, t)$, $\dot{\eta}(x, y, z, t)$, $\dot{\zeta}(x, y, z, t)$ for $t = 0$. Thus the initial conditions are specified by $3n(n+1)$ coefficients. These are, however, not entirely independent, for they must be determined so that (6.1) and (6.2) may be satisfied. Each of (6.1) and (6.2) gives $\frac{1}{2} n(n-1)$ relations between the coefficients in ξ, η, ζ and in $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ at $t=0$, and we have $2n^2 + 4n$ independent coefficients. On the other hand we have shown in section 3 that there are $2n^2 + 4n$ free oseillations of order n where $\lambda = 0$ is a multiple root. This does not however mean that the motions corresponding to the same λ are not independent but the multiple root simply introduces the same number of arbitrary constants as the multiplicity of the root. Thus the general solution of the equations of motion supplemented by the equation of continuity and the surface condition allow $2n^2 + 4n$ arbitrary constants while the initial condition is specified by $2n^2 + 4n$ independent coefficients.

Thus we have proved that is no ambiguity at all in expressing the motions of a liquid mass in terms of initial conditions.

7. The limiting form of the characteristic surface deformation.

The time constants of the surface oscillations are given as the roots of the equation (3.9). The pear-shaped deformation of a Jacobi ellipsoid enters through a surface deformationn which is represented by a Lamé function of order $n = 3$, and a method for calculating the characteristic frequencies for $n = 3$ has been given in Yabushita (1965). Specifically, the pear-shaped figure corresponds to the surface deformation represented the Poincaré polynomial denoted by Q_2. For explicit expressions for Poincaré polynomials in terms of (x, y, z) and (a, b, c), see Yabushita (1965). In terms of the Poincaré polynomials, the surface deformation, σ of order n is expressed as

$$\sigma = \Sigma A_k\, Q_k \tag{7.1}$$

where summation is over k from 1 to $2n + 1$ in general, and 7 in case of $n = 3$. The characteristic frequencies, λ's are then given as solutions of the equation

$$(\lambda^2 - 4\omega^2)\; A_k = 2\pi G \rho\, \Sigma h_{ik}\; (H_o - H_i) A_i, \quad k = 1, 2, \ldots (2n+1)\,. \tag{7.2}$$

where summation is over i from 1 to $2n+1$, in general, and to 7 for $n = 3$ (Yabushita 1965). Explicit expressions for the coefficients h_{ik} have been derived in Yabushita (1965).

Since some of the coefficients, h_{ik} vanish, the equation (7.2) splits into two parts, namely,

$$\left.\begin{aligned} &\sum_{i=1,6,7} G_{ik} A_i = 0, \\ &\sum_{i=2}^{5} G_{ik} A_i = 0, \end{aligned}\right\} \tag{7.3}$$

where

$$G_{ik} = 2\pi G\rho h_{ik}(H_o - H_i) - (\lambda^2 - 4\omega^2)\delta_{ik} \tag{7.4}$$

and where δ_{ik} is Kronecker's delta.

Now, as the point of bifurcation is approached, the frequency λ which changes sign at the point of bifurcation becomes small. It may be noted that the h's are of orders O (1), O ($1/\lambda$), and O ($1/\lambda^2$). It will be seen later that the h's of the order O ($1/\lambda$) determines the A's. Hence the term ($\lambda^2 - \omega^2$) will be neglected in the calculation to follow. The equations (7.3) may be written in the form

$$\left.\begin{aligned}
h_{22}(H_0-H_2)A_2+h_{32}(H_0-H_3)A_3+h_{42}(H_0-H_4)A_4+h_{52}(H_0-H_5)A_5&=0,\\
h_{23}(H_0-H_2)A_2+h_{33}(H_0-H_3)A_3+h_{43}(H_0-H_4)A_4+h_{53}(H_0-H_5)A_5&=0,\\
h_{24}(H_0-H_2)A_2+h_{34}(H_0-H_3)A_3+h_{44}(H_0-H_4)A_4+h_{54}(H_0-H_5)A_5&=0,\\
h_{25}(H_0-H_2)A_2+h_{35}(H_0-H_3)A_3+h_{45}(H_0-H_4)A_4+h_{55}(H_0-H_5)A_5&=0.
\end{aligned}\right\} \tag{7.5}$$

First, neglect the h's which involve $1/\lambda$ and take into account only terms which contain $1/\lambda^2$. The above equation then splits into two, namely;

$$h_{22}(H_0-H_2)A_2+h_{32}(H_0-H_3)A_3=0$$
$$h_{23}(H_0-H_2)A_2+h_{33}(H_0-H_3)A_3=0$$

and

$$h_{44}(H_0-H_4)A_4+h_{54}(H_0-H_5)A_5=0$$
$$h_{45}(H_0-H_4)A_4+h_{55}(H_0-H_5)A_5=0.$$

From the first set of equations, one gets $A_3 = 0$, A_2 = arbitrary, because $H_0 - H_2 = 0$ for $\lambda = 0$. Now the determinant of the coefficients of the second set of equations may be shown to be zero. In other words, the second set of equations yield arbitrary solutions for A_4 and A_5. This means that the second set of equations are inappropriate for determining the A's. We therefore consider the first two equations in (7.5) to determine A_4 and A_5. For non-zero solutions to exist, it is necessary and sufficient that

$$h_{42}h_{53} - h_{52}h_{43} \neq 0. \tag{7.6}$$

In order to prove the inequality, we proceed as follows.
We first note that

$$\lim_{\lambda\to 0} b_{4,5} = -\frac{c^2}{c^2 + h_{4,5}} \quad .$$

Then we have

$$\left.\begin{aligned}
-\frac{\lambda D_1}{2i\omega}h_{j2} &= A_jE_5 + B_jE_1\ ,\\
-\frac{\lambda D_1}{2i\omega}h_{j3} &= A_jE_6 + B_jE_2\ ,
\end{aligned}\right\} \quad j = 4,\ 5$$

where

$$\left.\begin{aligned} A_j &= \frac{3}{a^2}\frac{1}{b^2+h_j} - \frac{c^2}{a^2b^2}\frac{1}{c^2h_j} - \frac{2}{b^2}\frac{1}{a^2+h_j}, \\ B_j &= \frac{1}{a^2}\frac{1}{a^2+h_j} - \frac{c^2}{a^4}\frac{1}{c^2+h_j}. \end{aligned}\right\} \quad j = 4, 5.$$

We have

$$S \equiv -\frac{\lambda^2 D^2_1}{4\omega^2}(h_{42}h_{53} - h_{52}h_{43}) = (E_5E_2 - E_1E_6)\ (A_4B_5 - A_5B_4).$$

In order to prove (7.6) we only have to prove that neither $E_5\ E_2 - E_1E_6$ nor $A_4B_5 - A_5B_4$ is zero. We can easily show that

$$E_1E_6 - E_2E_5 = \frac{h_2h_3\ (h_2-h_3)(a^2-c^2)(b^2-c^2)(a^2-b^2)}{a^2b^2c^4\ (c^2+h_3)(c^2+h_2)(a^2+h_2)(a^2+h_3)(b^2+h_2)(b^2+h_3)} \neq 0,$$

$$A_4B_5 - A_5B_4 = \left(\frac{c^2}{a^6} + \frac{1}{a^4} + \frac{3c^2}{a^4b^2}\right)\frac{(a^2-b^2)(h_4-h_5)}{(a^2+h_4)(a^2+h_5)(c^2+h_4)(c^2+h_5)} \neq 0.$$

Thus we have proved the inequality (7.6).

Therefore the limiting form of the surface deformation is given by the initial member of the pear-shaped figures. The limiting form of the surface deformation, σ thus takes the form

$$\sigma\ (x, y, z, t) = \lim_{\lambda\to 0}\ (\alpha e^{i\lambda t} + \beta e^{-i\lambda t})\ L_2M_2N_2 +$$

deformations which oscillate before and beyond the point of bifurcation,

where α and β are arbitrary constants to be determined by initial conditions. Let us put

$$(\alpha e^{i\lambda t} + \beta e^{-i\lambda t})_{t=0} = A,$$

$$\frac{d}{dt}(\alpha e^{i\lambda t} + \beta e^{-i\lambda t})_{t=0} = \mathrm{B}.$$

Then in the limit $\lambda \to 0$, we have

$$\lim_{\lambda\to 0}\ (\alpha e^{i\lambda t} + \beta e^{-i\lambda t}) = A + Bt.$$

Thus, exactly at the point of bifurcation, the surface deformation σ develops with time t according to

$\sigma(x, y, z, t) = (A+Bt)\ L_2M_2N_2 +$ *deformations which oscillate before and beyond the point of bifureation,*

where A and B are now arbitrary constants to be determined by initial conditions.

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